

Semi-Symmetric Non-Metric Connection

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Abstract

Recently S. K. Chaubey and R. H. Ojha [1] introduced a semi-symmetric non-metric connection in almost contact metric manifold. The purpose of the present paper is to study this connection in a Sasakian manifold. We have also studied curvature tensors of a semi-symmetric non-metric T-connection in an almost contact metric manifold.

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1 Introduction

Let there exist an odd dimensional differentiable manifold M_n , $n = 2m + 1$, of differentiability class C^∞ , a C^∞ vector valued linear function F , a C^∞ vector field T and a C^∞ 1-form A satisfying

$$\overline{\overline{X}} + X = A(X)T, \quad (1)$$

$$A(\overline{X}) = 0, \quad (2)$$

where $\overline{X} \stackrel{def}{=} FX$, for arbitrary vector field X , then M_n is called an almost contact metric manifold and the structure (F, T, A) is called an almost contact

structure. In view of (1) and (2), we find

$$(a) \quad \overline{T} = 0, \quad (b) \quad A(T) = 1, \quad (c) \quad \text{rank}(F) = n - 1. \quad (3)$$

If a non-singular metric tensor g of type $(0, 2)$ satisfies

$$g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)A(Y), \quad (4)$$

for arbitrary vector fields X and Y , then an almost contact manifold M_n endowed with g is called an almost contact metric structure to M_n [2]. Putting T for X in (4) and using (3) (a) and (3) (b), we get

$$g(T, Y) = A(Y), \quad (5)$$

Put

$$'F(X, Y) \stackrel{\text{def}}{=} g(\overline{X}, Y), \quad (6)$$

then we have

$$'F(X, Y) + 'F(Y, X) = 0, \quad (7)$$

An almost contact metric manifold on which

$$D_X T = \overline{X}, \quad (8)$$

holds for arbitrary vector field X is called a K -contact Riemannian manifold. For a K -contact Riemannian manifold

$$(D_X A)(Y) = g(\overline{X}, Y), \quad (9)$$

If on a K -contact Riemannian manifold

$$(D_X 'F)(Y, Z) = A(Y)g(X, Z) - A(Z)g(X, Y) \quad (10)$$

holds, then the manifold is known as a Sasakian manifold [3]. Where D denotes the Riemannian connection.

2 Semi-symmetric non-metric connection

A linear connection $\tilde{B}$ on (M_n, g) defined as

$$\tilde{B}_X Y = D_X Y - A(Y)X - g(X, Y)T \quad (11)$$

for arbitrary vector fields X and Y , is said to be a semi-symmetric non-metric connection [1]. The torsion tensor $\tilde{S}$ of the connection $\tilde{B}$ and the metric tensor g are given by

$$\tilde{S}(X, Y) = A(X)Y - A(Y)X, \quad (12)$$

and

$$(\tilde{B}_X g)(Y, Z) = 2[A(Y)g(X, Z) + A(Z)g(X, Y)] \quad (13)$$

If in addition to (12) and (13)

$$(a) \quad \tilde{B}_X T = 0, \quad \text{or} \quad (b) \quad (\tilde{B}_X A)(Y) = 0 \quad (14)$$

hold for arbitrary vector fields X and Y , then the connection $\tilde{B}$ is said to be a semi-symmetric non-metric T -connection. Also, from (11) and (14), we have

$$D_X T - X - A(X)T = 0 \Leftrightarrow (D_X A)(Y) + g(X, Y) + A(X)A(Y) = 0 \quad (15)$$

Now, we put (11) as

$$\tilde{B}_X Y = D_X Y + H(X, Y), \quad (16)$$

where

$$H(X, Y) = -A(Y)X - g(X, Y)T \quad (17)$$

Let us define

$$(a) \quad \tilde{S}(X, Y, Z) \stackrel{\text{def}}{=} g(\tilde{S}(X, Y), Z) \quad (b) \quad {}'H(X, Y, Z) \stackrel{\text{def}}{=} g(\tilde{H}(X, Y), Z) \quad (18)$$

then we can write

$$(a) \quad \tilde{S}(X, Y, Z) = A(X)g(Y, Z) - A(Y)g(X, Z) \quad (19)$$

$$(b) \quad {}'H(X, Y, Z) = -A(Y)g(X, Z) - A(Z)g(X, Y)$$

Theorem 2.1 *Let $\tilde{B}$ be a semi-symmetric non-metric connection in a Sasakian manifold with a Riemannian connection D , then we have*

$$(a) \quad (\tilde{B}_X' F)(Y, \bar{Z}) = A(Y)[(D_X A)(\bar{Z}) - (D_X A)(Z)] \quad (20)$$

$$(b) \quad (\tilde{B}_X' F)(\bar{Y}, Z) = A(Z)[(D_X A)(Y) - (D_X A)(\bar{Y})]$$

Proof We have

$$\begin{aligned} X('F(Y, Z)) &= (D_X' F)(Y, Z) + 'F(D_X Y, Z) + 'F(Y, D_X Z) \\ &= (\tilde{B}_X' F)(Y, Z) + 'F(\tilde{B}_X Y, Z) + 'F(Y, \tilde{B}_X Z) \end{aligned}$$

With the help of (11), this equation becomes

$$(\tilde{B}_X' F)(Y, Z) = (D_X' F)(Y, Z) + 'H(X, Y, \bar{Z}) - 'H(X, Z, \bar{Y}), \quad (21)$$

Using (19) (b) in this equation, we obtain

$$(\tilde{B}_X' F)(Y, Z) = (D_X' F)(Y, Z) - A(Y)g(X, \bar{Z}) + A(Z)g(X, \bar{Y}), \quad (22)$$

From (9), (22) becomes

$$(\tilde{B}_X' F)(Y, Z) = (D_X' F)(Y, Z) + A(Y)(D_X A)(Z) - A(Z)(D_X A)(Y), \quad (23)$$

Barring Z on both sides and using (10) in (23), we have (20) (a). Also from (10) and (23), we find (20) (b).

Theorem 2.2 *Let $\tilde{B}$ be a semi-symmetric non-metric connection in a Sasakian manifold with a Riemannian connection D , then we have*

$$(\tilde{B}_{\bar{X}}'F)(Y, Z) + (\tilde{B}_X'F)(\bar{Y}, Z) + (\tilde{B}_X'F)(Y, \bar{Z}) = 0 \quad (24)$$

Proof Barring X, Y, Z in (23) respectively, we get

$$(a) \quad (\tilde{B}_{\bar{X}}'F)(Y, Z) = (D_{\bar{X}}'F)(Y, Z) + A(Y)(D_{\bar{X}}A)(Z) - A(Z)(D_{\bar{X}}A)(Y) \quad (25)$$

$$(b) \quad (\tilde{B}_X'F)(\bar{Y}, Z) = (D_X'F)(\bar{Y}, Z) - A(Z)(D_XA)(\bar{Y})$$

$$(c) \quad (\tilde{B}_X'F)(Y, \bar{Z}) = (D_X'F)(Y, \bar{Z}) + A(Y)(D_XA)(\bar{Z})$$

Adding these equations, we obtain

$$\begin{aligned} (\tilde{B}_{\bar{X}}'F)(Y, Z) + (\tilde{B}_X'F)(\bar{Y}, Z) + (\tilde{B}_X'F)(Y, \bar{Z}) &= (D_{\bar{X}}'F)(Y, Z) \\ &+ (D_X'F)(\bar{Y}, Z) + (D_X'F)(Y, \bar{Z}) + A(Y)(D_{\bar{X}}A)(Z) \\ &- A(Z)(D_{\bar{X}}A)(Y) - A(Z)(D_XA)(\bar{Y}) + A(Y)(D_XA)(\bar{Z}) \end{aligned}$$

Using (9) and (10) in the above equation, we get the result.

Theorem 2.3 *Let $\tilde{B}$ be a semi-symmetric non-metric connection in a Sasakian manifold with a Riemannian connection D , then we have*

$$(\tilde{B}_X'F)(Y, \bar{Z}) + (\tilde{B}_Y'F)(\bar{Z}, X) + (\tilde{B}_{\bar{Z}}'F)(X, Y) + 2'\tilde{S}(X, Y, Z) = 0$$

Proof From (23), we have

$$(a) \quad (\tilde{B}_X'F)(Y, \bar{Z}) = (D_X'F)(Y, \bar{Z}) + A(Y)(D_XA)(\bar{Z}) \quad (26)$$

Similarly,

$$(b) \quad (\tilde{B}_Y'F)(\bar{Z}, X) = (D_Y'F)(\bar{Z}, X) - A(X)(D_YA)(\bar{Z})$$

$$(c) \quad (\tilde{B}_{\bar{Z}}'F)(X, Y) = (D_{\bar{Z}}'F)(X, Y) + A(X)(D_{\bar{Z}}A)(Y) - A(Y)(D_{\bar{Z}}A)(X)$$

Adding these equations, we obtain

$$\begin{aligned} (\tilde{B}_X'F)(Y, \bar{Z}) + (\tilde{B}_Y'F)(\bar{Z}, X) + (\tilde{B}_{\bar{Z}}'F)(X, Y) &= (D_X'F)(Y, \bar{Z}) \\ &+ (D_Y'F)(\bar{Z}, X) + (D_{\bar{Z}}'F)(X, Y) + A(Y)(D_XA)(\bar{Z}) \\ &- A(X)(D_YA)(\bar{Z}) + A(X)(D_{\bar{Z}}A)(Y) - A(Y)(D_{\bar{Z}}A)(X) \end{aligned}$$

Using (9), (10) and (19) (a) in the above equation, we get the result.

Theorem 2.4 *In a Sasakian manifold with a semi-symmetric non-metric connection $\tilde{B}$, we have*

$$(\tilde{B}_XA)(\bar{Y}) = (D_XA)(\bar{Y}) - 'F(X, Y) \quad (27)$$

Proof Covariant derivative of (5) with respect to $\tilde{B}$ gives

$$(\tilde{B}_XA)(Y) = (\tilde{B}_Xg)(Y, T) + g(Y, \tilde{B}_XT) \quad (28)$$

Using (11) and (13), (28) becomes

$$(\tilde{B}_XA)(Y) = (D_XA)(Y) + g(Y, Z) + A(X)A(Y) \quad (29)$$

Barring Y in (29) and using (6), we get the result.

3 Curvature tensor of a semi-symmetric non-metric T -connection $\tilde{B}$

The curvature tensor of the connection $\tilde{B}$

$$R(X, Y, Z) = \tilde{B}_X \tilde{B}_Y Z - \tilde{B}_Y \tilde{B}_X Z - \tilde{B}_{[X, Y]} Z$$

and that of the connection D

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z$$

are related as [1]

$$R(X, Y, Z) = K(X, Y, Z) - \beta(X, Z)Y + \beta(Y, Z)X \quad (30)$$

$$-g(Y, Z)(D_X T - A(X)T) + g(X, Z)(D_Y T - A(Y)T),$$

where

$$\beta(X, Y) = (D_X A)(Y) + A(X)A(Y) + g(X, Y) \quad (31)$$

is a tensor field of type $(0, 2)$. In view of (14) (b) and (29), (31) gives

$$\beta(X, Y) = 0 \quad (32)$$

and hence in view of (31), (30) becomes

$$R(X, Y, Z) = K(X, Y, Z) - g(Y, Z)X + g(X, Z)Y \quad (33)$$

Contracting (33) with respect to X , we get

$$\tilde{Ric}(Y, Z) = Ric(Y, Z) - (n - 1)g(Y, Z), \quad (34)$$

where

$$\tilde{Ric}(Y, Z) \stackrel{\text{def}}{=} (C_1^1 R)(Y, Z) \quad \text{and} \quad Ric(Y, Z) \stackrel{\text{def}}{=} (C_1^1 K)(Y, Z)$$

are the Ricci-tensors with respect to the connections $\tilde{B}$ and D respectively. Again from (34), we have

$$\tilde{R}Y = RY - (n - 1)Y \quad (35)$$

and

$$\tilde{r} = r - n(n - 1), \quad (36)$$

where the Ricci operators $\tilde{R}$ and R of the connections $\tilde{B}$ and D are defined by

$$\tilde{Ric}(Y, Z) \stackrel{\text{def}}{=} g(\tilde{R}Y, Z) \quad ; \quad Ric(Y, Z) \stackrel{\text{def}}{=} g(RY, Z)$$

and the scalar curvature tensor $\tilde{r}$ and r of $\tilde{B}$ and D are

$$\tilde{r} \stackrel{\text{def}}{=} \text{trace}(\tilde{R}) \quad ; \quad r \stackrel{\text{def}}{=} \text{trace}(R)$$

respectively.

Theorem 3.1 *If an almost contact metric manifold M_n equipped with a semi-symmetric non-metric T -connection $\tilde{B}$ whose Ricci-tensor vanishes, then the curvature tensor of $\tilde{B}$ coincides with the Weyl-projective curvature tensor of the manifold.*

Proof If $\tilde{Ric}(Y, Z) = 0$, then (34) gives

$$Ric(Y, Z) = (n - 1)g(Y, Z) \quad (37)$$

The Weyl projective curvature tensor of the Riemannian connection D [2] is

$$W(X, Y, Z) = K(X, Y, Z) - \frac{1}{n - 1}[Ric(Y, Z)X - Ric(X, Z)Y] \quad (38)$$

Using (33) and (37) in (38), we have

$$W(X, Y, Z) = R(X, Y, Z).$$

Corollary 3.1 *If an almost contact metric manifold M_n equipped with a semi-symmetric non-metric T -connection $\tilde{B}$ whose curvature tensor vanishes, then the manifold is projectively flat.*

Theorem 3.2 *Let M_n be an almost contact metric manifold admitting a semi-symmetric non-metric T -connection $\tilde{B}$ whose Ricci-tensor vanishes, then the curvature tensor with respect to the semi-symmetric non-metric T -connection coincides with the conformal curvature tensor of the manifold.*

Proof The conformal curvature tensor V of the Riemannian connection D is [2]

$$\begin{aligned} V(X, Y, Z) = & K(X, Y, Z) - \frac{1}{(n - 2)}[Ric(Y, Z)X - Ric(X, Z)Y + g(Y, Z)RX \\ & - g(X, Z)RY] + \frac{r}{(n - 1)(n - 2)}[g(Y, Z)X - g(X, Z)Y] \end{aligned} \quad (39)$$

In consequence of (33), (34), (35) and (36), (39) gives

$$V(X, Y, Z) = R(X, Y, Z)$$

Corollary 3.2 *If an almost contact metric manifold M_n , the curvature tensor of a semi-symmetric non-metric T -connection $\tilde{B}$ vanishes, then it is conformally flat.*

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