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SOME PROPERTIES OF SASAKIAN MANIFOLDS ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. The object of the present work is to study generalized Ricci-recurrent Sasakian manifolds admitting a quarter-symmetric metric connection. We also introduce and study ϕ-Ricci semi-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection.

Introduction. The idea of metric connection with torsion in a Riemannian manifold was introduced by Hayden (1932). Further some properties of semi-symmetric metric connection has been studied by Yano (1970). Agashe et. al. (1992) defined and studied a semi-symmetric non-metric connection in a Riemannian manifold. This was further developed by many geometers. In 1975 Golab defined and studied quarter-symmetric connection in a differentiable manifold with affine connection, which generalizes the idea of semi-symmetric connection. After Golab quarter symmetric connection has been studied by many geometers like as Rastogi (1978, 1987), Mishra et. al., (1980), Yano et. al., (1982) and other. In a recent paper Mondal et. al., (2009), studied some properties of a quarter-symmetric metric connection on a Sasakian manifold.

The concept of generalized Ricci-recurrent manifold was introduced by De et. al., (1991). Generalized Ricci-recurrent manifold was studied by many geometers. The notion of Ricci semi-symmetric Riemannian manifold was introduced by Deszcz et. al., (1989). In the present paper we study generalized Ricci-recurrent Sasakian manifolds admitting a quarter-symmetric metric connection. We also introduce and study ϕ-Ricci semi-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection.

The paper is organized as follows: In section 2, we give brief introduction about Sasakian manifolds. In section 3, we give some formulae for quarter-symmetric metric connection which we use later. In section 4, we study generalized Ricci-recurrent Sasakian manifolds admitting a quarter-symemtric metric connection. We prove that if a generalized Ricci-recurrent Sasakian manifold admits a quarter-symmetric metric connection, then $B = -2A$. In the last section we have introduced the notion of ϕ-Ricci semi-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection and show that a Sasakian manifold is ϕ-Ricci semi-symmetric with respect to Riemannian connection ∇ if and only if it is ϕ-Ricci semi-symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$.

2. Preliminaries. Let M be an $n = (2m+1)$-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field

ϕ, a vector field ξ, a 1-form η and a Riemannian metric g. Then

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \forall X, Y \in TM. \tag{2.2}$$

From (2.1) and (2.2), it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \ g(X, \xi) = \eta(X), \ \forall X, Y \in TM. \tag{2.3}$$

An almost contact metric manifold M is said to be

(a) a contact metric manifold if

$$g(X, \phi Y) = d\eta(X, Y), \forall X, Y \in TM; \tag{2.4}$$

(b) a K-contact manifold if the vector field ξ is killing equivalently

$$\nabla_X \xi = -\phi X, \tag{2.5}$$

where ∇ is Riemannian connection and

(c) a Sasakian manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \ \forall X, Y \in TM. \tag{2.6}$$

A K-contact manifold is a contact metric manifold, while converse is true if the Lie derivative of ϕ in the characteristic direction ξ vanishes. A Sasakian manifold is always K-contact manifold. A 3-dimensional K-contact manifold is a Sasakian manfiold.

It is well known that a contact metric manifold is Sasakian if and only if

$$R(X, Y, \xi) = \eta(Y)X - \eta(X)Y, \forall X, Y \in TM. \tag{2.7}$$

In a Sasakian manifold equipped with the structure (ϕ, ξ, η, g), the following relations also hold :

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{2.8}$$

$$R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.10}$$

$$R(X, \xi, Y) = \eta(Y)X - g(X, Y)\xi, \tag{2.11}$$

$$\eta(R(X, Y, Z)) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.12}$$

for all $X, Y, Z \in TM$, where S is the Ricci-tensor. For more details we refer to (Blair, 1976; Sasaki, 1975 and Yano and Kon, 1984).

3. Quarter-Symmetric metric connection. Here we consider a quarter symmetric metric connection $\tilde{\nabla}$ on a Sasakian manifold given by (Mondal and De, 2009)

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y \tag{3.1}$$

A relation between the curvature tensor $\tilde{R}(X,Y,Z)$ of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and $R(X,Y,Z)$ of M with respect to the Riemannian connection ∇ is given by

$$\begin{aligned}\tilde{R}(X,Y,Z) = R(X,Y,Z) - 2d\eta(X,Y)\phi Z + \eta(X)g(Y,Z)\xi \\ -\eta(Y)g(X,Z)\xi + \{\eta(Y)X - \eta(X)Y\}\eta(Z).\end{aligned} \tag{3.2}$$

Also from (3.2) we obtain

$$\tilde{S}(Y,Z) = S(Y,Z) - 2d\eta(\phi Z, Y) + g(Y,Z) + (n-2)\eta(Y)\eta(Z), \tag{3.3}$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇ respectively. From (3.3) it is clear that in a Sasakian manifold the Ricci tensor with respect to the quarter-symmetric metric connection is symmetric.

4. Generalized Ricci-recurent Sasakian manifolds admitting a quarter-symmetric metric connection. The notion of generalized Ricci-recurrent manifolds was introduced by De et. al., (1991).

A non-flat n-dimensional differentiable manifold $M(n > 3)$ is called generalized Ricci recurrent if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + (n-1)B(X)g(Y,Z) \tag{4.1}$$

where ∇, in the Riemannian connection and A and B are two 1-forms (B is non-zero) and are defined by $A(x) = g(x,P)$, $B(x) = g(x,Q)$ and P and Q are vector fields related to the 1-forms $A = B$ respectively. Here we give the following definition.

A non-flat n-dimensional differentiable manifold $M(n > 3)$ is called generalized Ricci-current with respect to quarter-symmetric metric connection if its Ricci tensor $\tilde{S}$ satisfies the condition

$$(\tilde{\nabla}_X \tilde{S})(Y,Z) = A(X)\tilde{S}(Y,Z) + (n-1)B(X)g(Y,Z) \tag{4.2}$$

where $\tilde{\nabla}$ is as previous section and $\tilde{S}$ is the Ricci tensor of quarter-symmetric metric connection $\tilde{\nabla}$. Now we prove the following theorem:

THEOREM 4.1 *Let M be a generalized Ricci-recurrent Sasakian manifold admitting a quarter-symmetric metric connection. Then $B = -2A$.*

Proof: Let M be a generalized Ricci-recurrent Sasakian manifold admitting a quarter-symmetric metric connection. So the equation (4.2) holds. Putting $Z = \xi$ in the equation (4.2), we have

$$(\tilde{\nabla}_X \tilde{S})(Y,\xi) = A(X)\tilde{S}(Y,\xi) + (n-1)B(X)\eta(Y). \tag{4.3}$$

After using the relations (3.3), (2.1), (2.10) the above equation gives

$$(\tilde{\nabla}_X \tilde{S})(Y,\xi) = (n-1)[2A(X) + B(X)]\eta(Y). \tag{4.4}$$

Now

$$(\tilde{\nabla}_X \tilde{S})(Y,\xi) = \tilde{\nabla}_X \tilde{S}(Y,\xi) - \tilde{S}(\tilde{\nabla}_X Y,\xi) - \tilde{S}(Y,\tilde{\nabla}_X \xi). \tag{4.5}$$

By virtue of equations (3.1),(2.1), (3.3), (2.10), (2.5) the above equation assumes the form

$$(\tilde{\nabla}_X \tilde{S})(Y,\xi) = S(\phi X, Y) + (1-2n)g(\phi X, Y). \tag{4.6}$$

Equating the right hand side of the equations (4.4) and (4.6) we get

$$(n-1)[2A(X) + B(X)]\eta(Y) = S(\phi X, Y) + (1-2n)g(\phi X, Y). \tag{4.7}$$

Putting $Y = \xi$ in the above equation and then making use of relations (2.1), (2.3) and (2.10), it follows that

$$(n-1)[2A(X) + B(X)] = 0, \tag{4.8}$$

which implies (since $n > 3$)

$$B(X) = -2A(X),$$

for arbitrary vector fields X; which proves the theorem.

5. ϕ-Ricci semi-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection. An n-dimensional Riemannian manifold M is called Ricci semi-symmetric (Deszcs and Hotlas, 1989) if

$$(R(X,Y)\cdot S)(Z,W) = 0, \tag{5.1}$$

for all X, Y, where $R(X,Y)$ denotes the curvature operator; which is equivalent to

$$S(R(X,Y,Z),W) + S(Z,R(X,Y,W)) = 0. \tag{5.2}$$

It is easy to see that if $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal basis of vector field in M, then $\{\phi e_1, \phi e_2, \ldots, \phi e_n\}$ is also a local orthonormal basis. So we can consider the following definition.

An n-dimensional Riemannian manifold M is called ϕ-Ricci semi-symmetric admitting a quarter-symmetric metric connection if

$$\tilde{S}(\tilde{R}(\phi X, \phi Y, \phi Z), \phi W)\tilde{S}(\phi Z, \tilde{R}(\phi X, \phi Y, \phi W)) = 0, \tag{5.3}$$

where $\tilde{R}(X,Y)$ is the curvature operator with respect to quarter-symmetric metric connection and $\tilde{S}$ is as the previous section. Now we prove the following theorem :

THEOREM 5.1 *A Sasakian manifold is ϕ-Ricci semi-symmetric with respect to Riemannian connection ∇ if and only if it is ϕ-Ricci semi-symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$.*

Proof: Consider the expression

$$\tilde{S}(\tilde{R}(X,Y,Z)W) + \tilde{S}(Z, \tilde{R}(X,Y,W)). \tag{5.4}$$

Putting the value of $\tilde{R}$ from equation(3.2) in the above equation and making use of relation (2.3) we have

$$\begin{aligned}
&\tilde{S}(\tilde{R}(X,Y,Z),W) + \tilde{S}(Z,\tilde{R}(X,Y,W))\\
&= \tilde{S}(R(X,Y,Z),W) + \eta(X)g(Y,Z)\tilde{S}(\xi,W)\\
&-\eta(Y)g(X,Z)\tilde{S}(\xi,W) + \eta(Y)\eta(Z)\tilde{S}(X,W)\\
&-\eta(X)\eta(Z)\tilde{S}(Y,W)\\
&+\tilde{S}(Z,R(X,Y,W)) + \eta(X)g(Y,W)\tilde{S}(Z,\xi)\\
&-\eta(Y)g(X,W)\tilde{S}(Z,\xi) + \eta(Y)\eta(W)\tilde{S}(Z,X)\\
&-\eta(X)\eta(W)\tilde{S}(Z,Y).
\end{aligned} \tag{5.5}$$

Now putting the value of $\tilde{S}$ from equation (3.3) in the above equation by virtue of (2.10), (2.2), (2.12) and skew-symmetric property of R, we get

$$\begin{aligned}
&\tilde{S}(\tilde{R}(X,Y,Z)W + \tilde{S}(Z,\tilde{R}(X,Y,W))\\
&= S(R(X,Y,Z),W) + S(Z,R(X,Y,W))\\
&+(3n-2)\{g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)\\
&+g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)\}\\
&+\{\eta(Y)\eta(Z)S(X,W) - \eta(X)\eta(Z)S(Y,W)\\
&+\eta(Y)\eta(W)S(X,Z) - \eta(X)\eta(W)S(Y,Z)\}.
\end{aligned} \tag{5.6}$$

Now replacing X by $\phi X, Y$ by $\phi Y, Z$ by ϕZ and W by ϕW respectively in the above equation by virtue of (2.1), it follows that

$$\begin{aligned}
&\tilde{S}(\tilde{R}(\phi X,\phi Y,\phi Z),\phi W) + \tilde{S}(\phi Z,\tilde{R}(\phi X,\phi Y,\phi W))\\
&= S(R(\phi X,\phi Y,\phi Z),\phi W) + S(\phi Z, R(\phi X,\phi Y,\phi W)).
\end{aligned} \tag{5.7}$$

The theorem at once follows from the equation (5.7).

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